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A HELICOPTER SUBMARINE SEARCH GAME

by

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This thesis examines a two-person zero sum game where a submarine, after revealing his position by causing a 'flaming datum', is hunted by a helicopter which arrives on the scene after a time delay. Various helicopter and submarine strategies are explored and simulation runs are used to determine the detection probability (payoffs) for each combination of helicopter and submarine strategy. The value of the game (detection probability) with the related optimal strategies is then obtained using linear programming. A modified random search equation is also derived using probabilities of detection obtained from different combinations of parameters used in the game. Similar and related games are also discussed with emphasis on the differences in assumptions made and approaches taken in order to solve the problem.

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A Helicopter Submarine Search Game

by

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ABSTRACT

This thesis examines a two-person zero sum game where a submarine, after revealing his position by causing a 'flaming datum', is hunted by a helicopter which arrives on the scene after a time delay. Various helicopter and submarine strategies are explored and simulation runs are used to determine the detection probability (payoffs) for each combination of helicopter and submarine strategy. The value of the game (detection probability) with the related optimal strategies is then obtained using linear programming. A modified random search equation is also derived using probabilities of detection obtained from different combinations of parameters used in the game. Similar and related games are also discussed with emphasis on the differences in assumptions made and approaches taken in order to solve the problem.

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I. INTRODUCTION

A. DESCRIPTION OF PROBLEM

A submarine has just fired a torpedo at a ship, scoring a direct hit. This causes the ship to stop immediately and be in flames, hence the term, 'flaming' datum. The ship on being hit immediately sends out distress calls and a sonar carrying helicopter responds by proceeding towards the 'flaming' datum. It arrives on scene after a time delay and thereafter makes repeated active sonar dips within the expanding farthest-on-circle(FOC). The term FOC is a commonly used search term which describes all the possible positions within a circle (of area $\pi U^2 t^2$), that a target can be, given a top speed of U. There is also a time delay for each dip due to the time required for lowering and raising the sonar transponder. The submarine on detecting each and every ping emitted by the helicopter's sonar reacts so as to minimize his probability of being detected within the helicopter's limited time on station. The game can only end in two ways; the submarine is found within the helicopter's sonar range during one of its dips and hence is detected or the submarine is undetected during the limited mission time of the helicopter. We consider an abstract version of this situation characterized by five parameters:

- V - Speed of helicopter
- U - Speed of submarine
- D - Time delay in dipping sonar
- τ_e - Time for helicopter to reach edge of FOC
- R - Radius of sonar detection system

B. BACKGROUND

Search theory in military applications has usually concentrated on the idea of attempting to find an object by moving a sensor sufficiently close to it. The object has a definite position in space and the distance between it and the sensor is crucial. Search theory has evolved from searching for stationary targets, to moving targets and presently, to evasive targets. Stationary and moving target problems are normally classified as one sided problems because the target does not know or make use of the knowledge of where the searcher is. The present emphasis is on two-sided problems,

which are more difficult to solve. These are problems where targets wish to avoid detection , and have the capability to do so.

The theory of Games is naturally applicable here since this is the situation of a maximizing player (searcher) wishing to maximize a payoff (for example, the probability of detecting the target) and a minimizing player (evader) wishing to do the opposite. In such situations, optimal strategies of both players are sought after as well as the value of the game where we want to know how best one side can deny or allow the other side an expected minimum or maximum payoff. One classical two sided game formulated by Morse and Kimbal [Ref. 1] involves searching for a submarine that is transiting a channel of varying width that is too long to allow the submarine to remain submerged during the entire passage. In his book, 'Geometric Games and their Applications' [Ref. 2] Ruckle considers a class of two-sided geometric games. Gal considers two sided search games played on networks [Ref. 3]. A good reference for bibligographies can be found in Dobbie's survey of Search Theory [Ref. 4]. This is updated by Washburn's tutorial on search theory where he covers the development of search theory up to 1986 [Ref. 5].

The author is interested in the submarine evasion game first considered by Danskin in which a submarine, after revealing his position by causing a 'flaming' datum, is hunted by a helicopter which arrives at the datum after a time delay [Ref. 6]. This is a special class of two sided search games where the target has the advantage of observing the searcher's action, but the searcher has the advantage of speed. This advantage will be slowly lost as search progresses since the various positions that the submarine can occupy will expand with time. Related games are the one-dimensional helicopter submarine games studied by Meinardi [Ref. 7] and more recently by Baston and Bostock [Ref. 8]. Meinardi solves a discrete form of the game while Baston and Bostock solve the continuous case. Besides Danskin's game, not much work has been done on the two dimensional case except the work by Thomas and Washburn [Ref. 9] in their paper on dynamic search games.

C. APPROACHES

Three solution approaches will be considered:

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1. Analytical

In the next chapter , we will see solutions to related games. The one dimensional 'flaming' datum problem can be solved either by using the methods of Meinardi or Baston and Bostock. However, the one dimensional case is only a special case where the search area can be considered to be a long channel with respect to the sonar detection radius. Attempts to solve the two dimensional case have not been clearly successful since additional simplifying assumptions are needed. In Danskin's game, the assumptions made (he assumes passive search) are clearly to the advantage of the helicopter. However, in Thomas and Washburn's dipping sonar game, two major assumptions are made, one that favours the submarine (nimbleness assumption) and another that does not (remain motionless between dips).

2. Data Analysis

Another way to solve the 'flaming' datum problem is to collect and analyse real world data. Such data may or may not be obtained from past wars, conflicts, or even field exercises. However, such data are few, expensive and difficult to obtain. A compromise is to conduct a two sided search experiment as was done by Washburn in his Expanding Area Search Experiment. [Ref. 10] However, Washburn's experiment differs from our 'flaming' datum problem. The target is assumed to be 'blind' with no knowledge of the searcher's position once the search commences while the searcher is assumed to be a ship or aeroplane continuously sweeping an area. In his experiment, different groups of NPS students are used as game players. This technique can perhaps be used as the next stage in our attempt to solve the 'flaming' datum problem. The only disadvantage is that the experiment will take a long time to complete since many different people are needed.

3. Simulation Methodology

The approach taken here is to build multiple simulation models where Monte Carlo runs are used to determine the detection probability when a helicopter uses a certain strategy against a certain submarine strategy . Two person zero sum solution methodology is then used to solve the resulting matrix game. This simple method allows one to estimate the outcome quickly and rather cheaply as compared to conducting numerous fleet exercises. It may also act as a starting point for more sophisticated models. A more detailed description will be given in the later chapters.

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II. RELATED GAMES

In this chapter, two one-dimensional and two two-dimensional games related to the 'flaming datum' problem will be discussed. These are the games by Meinardi [Ref. 7], Baston and Bostock [Ref. 8], Danskin [Ref. 6] and Thomas and Washburn [Ref. 9] respectively. The 'flaming' datum problem can only be formulated as a one dimensional game (in space) if the search area under consideration involves a long channel such that the sonar detection sweep width is at least greater than the channel width. Meinardi [Ref. 7] has solved a discrete (in time) version of such a game while more recently, Baston and Bostock [Ref. 8] have solved the continuous version. In all the two dimensional cases treated so far, many assumptions have to be made in order to solve the game. This is evident in both Danskin's and Thomas and Washburns' games. General analytical techniques for the two-dimensional game are presently unavailable.

A. ONE DIMENSIONAL GAMES

1. A Sequentially Compounded Search Game

Meinardi [Ref. 7] considers a multi-staged search game where the target is hiding in a row of n_0 boxes labelled 1 to n_0. At each stage , the searcher selects a box and examines it. The probability of finding the target when the correct box is searched is equal to $q \leq 1.0$. If the target is not found, the target may either move to a neighbouring box or remain in the same box, and the next stage is played. The target as well as the searcher keeps track of which boxes have been searched. The searcher is limited only in the number of boxes to be searched, so that the time taken to search a box or transit between boxes does not come into consideration.
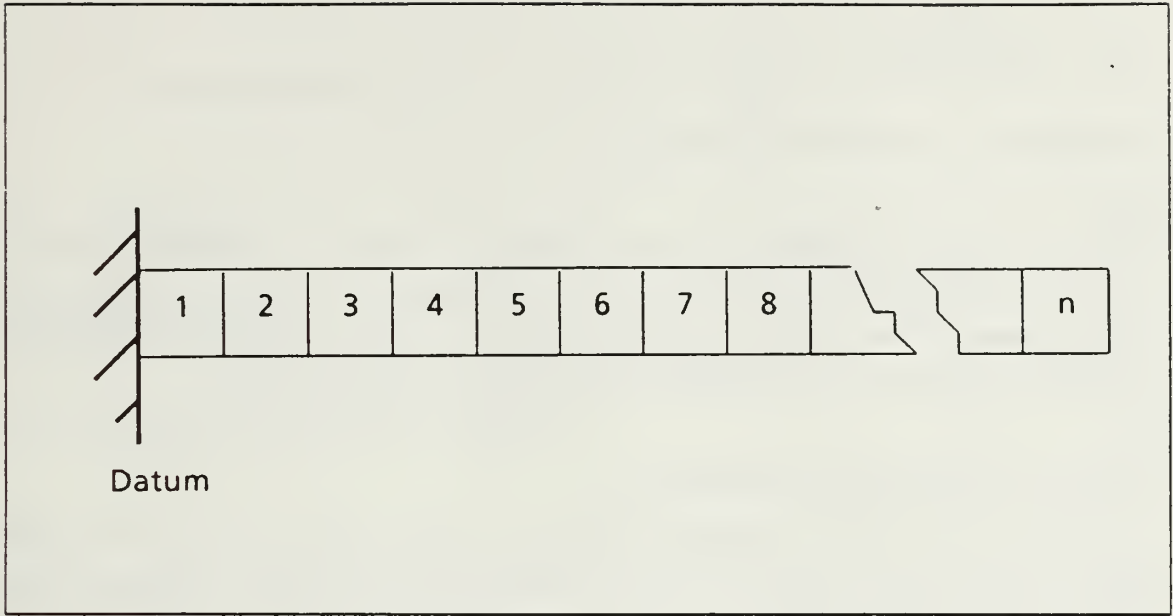
Figure 1.   Boxes for Meinardi's game

At the start of the game, the target is in box 1, (see Figure 1) corresponding to the datum. A certain time delay may occur before the first search is made. The target can therefore be in any of the first n_0 boxes when the first search is made. With each stage, the number of boxes which may contain the target is augmented by one (since the target is limited by jumping to neighbouring boxes only). Meinardi [Ref. 7] shows that the optimal strategies of the searcher and the target are such that they alway attempt to distribute themselves as uniformly as possible over the number of boxes available. He also shows that if q exceeds a certain critical value q_{crit}, the target will not be able to use such a strategy. In such a case, the target will attempt to distribute his positional probability over whatever boxes that he can.

Meinardi derived the critical q to be

$$q_{crit} = \frac{n+1-k}{k}$$

where k is the box that is searched when n boxes are available. If $q \leq q_{crit}$, the target can equalize its probability of being in any of the $(n+1)$ boxes available in the next stage. Since the smallest value of q_{crit} is with $k = n$, the smallest q_{crit} in the game (denoted $\hat{q}_{crit}$) is given by

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$$\hat{q}_{crit} = \frac{1}{n_0 + s_0 - 2},$$

except that $\hat{q}_{crit} = 1$ for a one stage game.

Here n_0 is the initial number of boxes available at the start of the game and s_0 is the number of stages available in the game. It is also useful for us here to reproduce some of his results. Case 1 shows the result for the case when the critical q value is not exceeded while case 2 is an example of a case when q is greater than $\hat{q}_{crit}$

Case 1: Two stage game, for a small q.

Let $\Gamma_{(n)}^{s}$ be the value of the game where $s+1$ is the number of stages of the game remaining and n, the number of boxes available (similar notations are also used for the optimal strategies $\overline{X}$ and $\overline{Y}$).

The value of the game is given by

$$\Gamma_{(n_0)}^{1} = (\frac{q}{n_0}) + (1 - \frac{q}{n_0})\ \Gamma_{(n_0-1)}^{0}\ \ where\ \ \Gamma_{(n_0-1)}^{0} = (\frac{q}{n_0+1}) \tag{2.1}$$

with optimal strategies

$$\overline{X}_{(n_0)}^{1} = \overline{Y}_{(n_0)}^{1} = [\ \frac{1}{n_0},\frac{1}{n_0},.....,\frac{1}{n_0}\ ] \tag{2.2}$$

$$\overline{X}_{(n_0-1)}^{0} = \overline{Y}_{(n_0-1)}^{0} = [\ \frac{1}{n_0+1},\frac{1}{n_0+1},...,\frac{1}{n_0+1}\ ] \tag{2.3}$$

where $\overline{X}$ and $\overline{Y}$ represent the distributions of searcher and target positions. In other words, the searcher is equally likely to search any boxes, and the target moves in such a manner that he is equally likely to be in any box. In this case, $\hat{q}_{crit} = \frac{1}{n_0}$ so the above results are valid as long as $q \leq \frac{1}{n_0}$.

The case when $q > \hat{q}_{crit}$, is more complicated. We will only show the results for the case of a two stage game where $n_0 = 2$ (as solved by Meinardi). More detailed derivation can be found in [Ref. 7].

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Case 2: Two stage game, $q > \frac{1}{2}$, $n_0 = 2$, $\hat{q}_{crit} = \frac{1}{2}$,

$$\Gamma_{(2)}^1 = (\frac{7-2q}{9-2q})q \tag{2.4}$$

with optimal strategies

$$\overline{X}_{(2)}^1 = [\ \frac{3}{9-2q}\ ,\ \frac{6-2q}{9-2q}\ ]\ \textit{for the searcher} \tag{2.5}$$

$$\overline{Y}_{(2)}^1 = [\ \frac{4}{9-2q}\ ,\ \frac{5-2q}{9-2q}\ ]\ \textit{for the target} \tag{2.6}$$

The above result shows that both the searcher and target do not equalize their probability of being in any of the boxes available. This happens because the target is unable to do so. To illustrate this, consider the case when $q = 1$ and where box 2 is searched in the first stage of the game. Clearly, the target must be in box 1 in the first stage or else it will be detected. The target is thus unable to move to box 3 at the second stage since it is limited to moving to neighbouring boxes only. The target can therefore only move to box 2 or remain in box 1 . If q is less than 1, then there will be some probability that the target will not be detected even if the correct box is selected. If the target is initially in box 2. then there would then be some probability of the target moving to box 3 in the second stage, but not as much as $\frac{1}{3}$ if $q > \hat{q}_{crit}$. In general. the target should equalize, as well as it can, the probability of being in any box.

2. A Helicopter-Submarine Game On The Real Line

Baston and Bostock [Ref. 8] also consider a game very similar to the flaming datum problem. Their game can be considered to be a continuous version of Meinardi's game [Ref. 7] since it is again a one-dimensional problem where the submarine is assumed to be moving in a long narrow channel. In their game, there is one helicopter carrying j anti-submarine bombs. The helicopter (max. velocity V) wishes to destroy the mobile submarine (max. velocity U) using its bomb. Each bomb has the same destructive radius R and there is a time lag D between the release of the bomb and the

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bomb exploding. The number of bombs is analogous to the number of dips available and the bomb's destruction radius is analogous to the sonar radius.
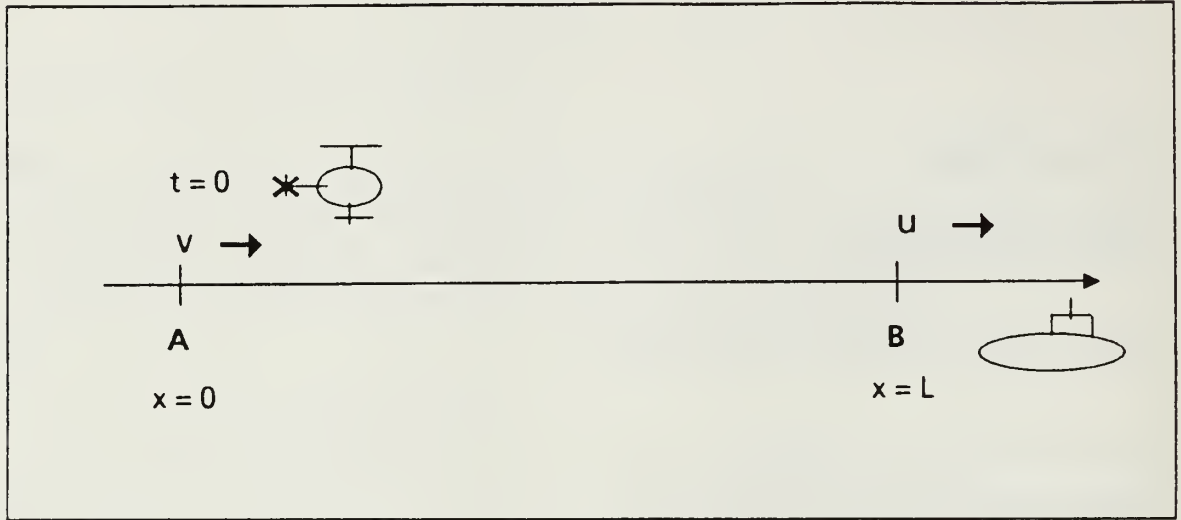
Figure 2. One dimensional helicopter-submarine on the real line .

The notation used in their game is illustrated in Figure 2 . The helicopter and submarine are initially at A and B respectively, distance L apart. If the bomb explodes within distance R from the submarine then the payoff to the helicopter is 1 unit; otherwise, he gets zero. Each player knows the initial position of the other player but not their subsequent positions. They are restricted to move at their maximum speed and instantaneous changes in velocities are allowed. The value of the game, Γ is given below:

(1) When $\dfrac{U(L + VD)}{V} \leq R$, the value is $\Gamma = 1$ regardless of the number of bombs that the helicopter is carrying since only one bomb is needed. The helicopter proceeds to point B at its maximum speed and drops his bomb there. The submarine could not travel a distance greater than R, given the time the helicopter takes to get to point B, $\dfrac{L}{V}$, plus the bomb activation time D.

(2) When $\dfrac{U(L + VD)}{V} > R$, the value of the game is given by

$$\Gamma = (\frac{j}{k}) , \qquad \text{for } 1 < j \leq k \text{ and}$$

$$\Gamma = 1, \qquad \text{for } j > k$$

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where k is the unique integer greater or equal to 2 which satisfies

$$\frac{U(L + VD)}{(k - 1)(V - U) + V} \leq R \leq \frac{U(L + VD)}{(k - 2)(V - U) + V} \qquad (2.7)$$

When $(j \geq 2)$ bombs are available to the helicopter, an additional condition is required in that the time delay must be small , that is, $R \geq UD - L$ The other case where $R < UD - L$ is still being studied by Baston and Bastock.

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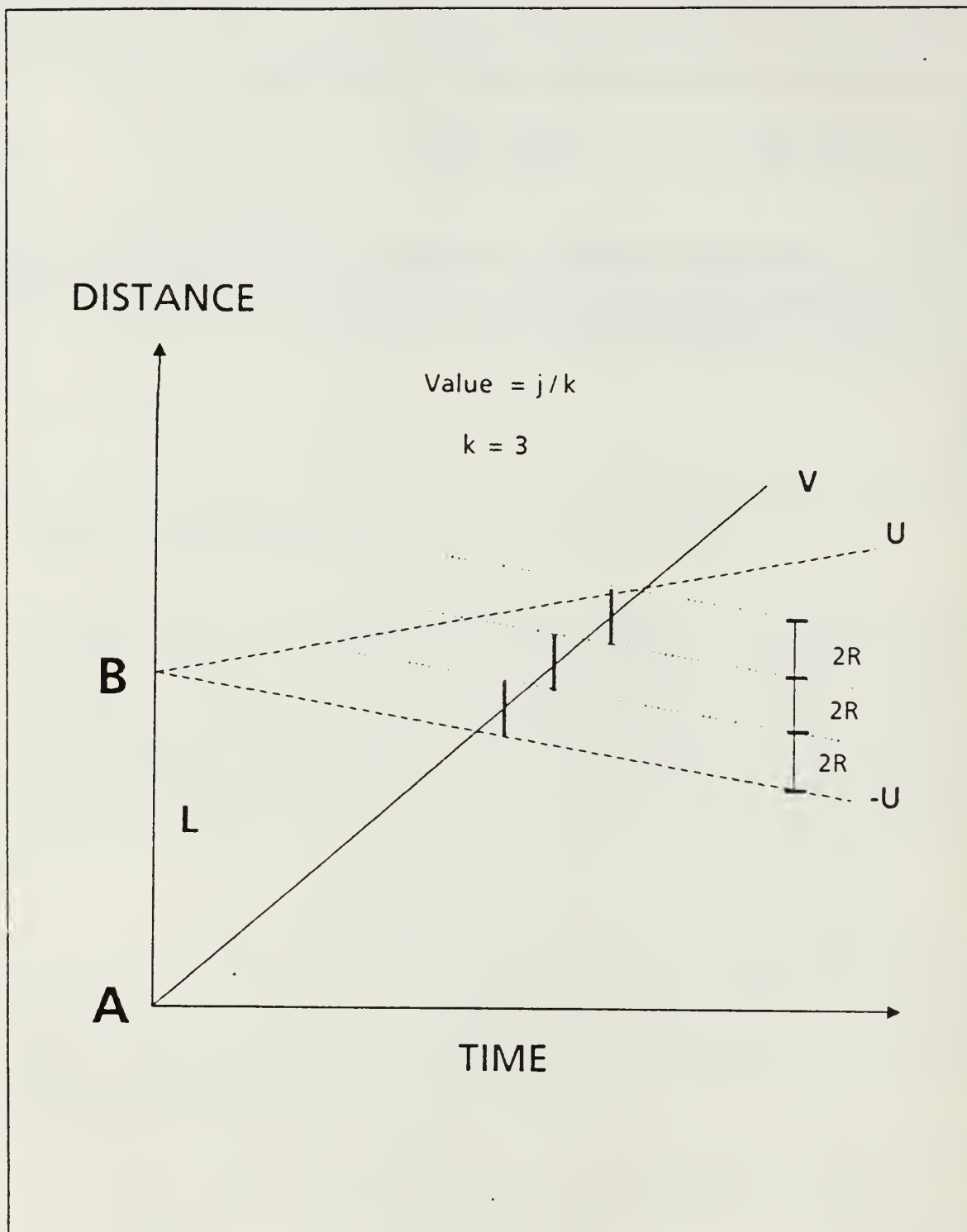
Figure 3.   Graphical representation of Baston and Bostock

A graphical representation of the game is shown in Figure 3 . The dashed lines show the possible positions of the submarine from B. The vertical short solid lines show where the bombs should be dropped. Note that a feasible submarine track must intersect at least one of those intervals. In the above example, $k = 3$ and the value of the game is ($\frac{j}{3}$). Hence, if $j = 3$, the value of the game is 1 and the submarine is always destroyed.

B. TWO DIMENSIONAL GAMES

1. Helicopter Versus Submarine Search Game

Danskin [Ref. 6] considers the case where the submarine has only 1 set of strategies, i.e., choose a fixed course θ ($0 \leq \theta \leq 360^0$) and speed U ($0 \leq U \leq U_{max}$) and adhere to them throughout the duration of the game. He assumes that the submarine knows nothing useful about what the helicopter is doing and therefore sees no reason for any changes in course or speed once selected. Danskin describes this strategy as the choice of a point in the submarine's 'speed circle' of radius U_{max} (See Figure 4). The submarine picks this point and stays at it throughout the game. With each dip, the helicopter's sonar detection system will cover an area of πR^2 in real space or an equivalent area of $\frac{\pi R^2}{t^2}$ in the speed circle at time t (see Figure 4 overleaf). He assumes that the helicopter strategies are to dip at a succession of points in the speed circle such that the corresponding covered circles do not overlap each other. Hence if the dips are at times $t_1, t_2, \ldots, t_n$. the helicopter will cut a total area $A_0 = \pi R^2 (\frac{1}{t_1^2} + \frac{1}{t_2^2} + \ldots \frac{1}{t_n^2})$ out of the speed circle. This is shown in Figure 4 overleaf.
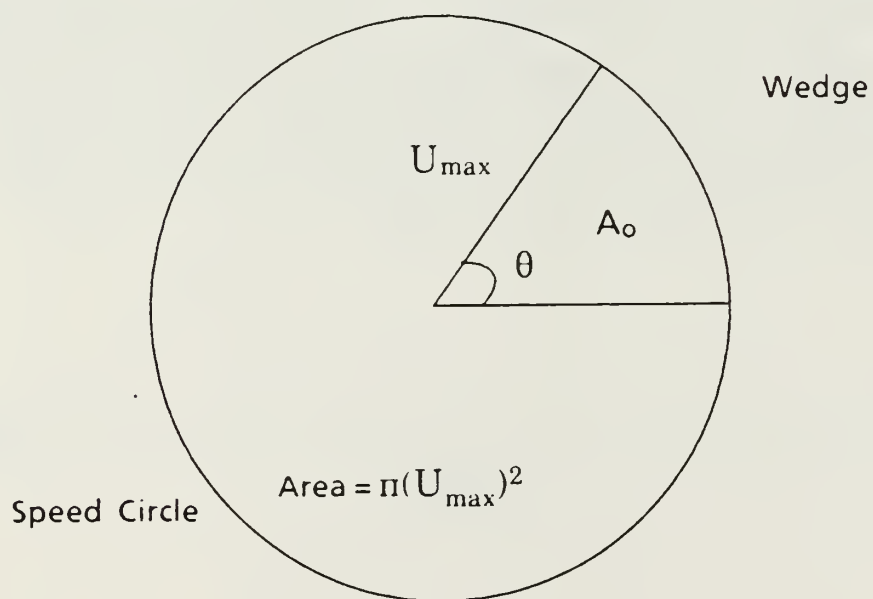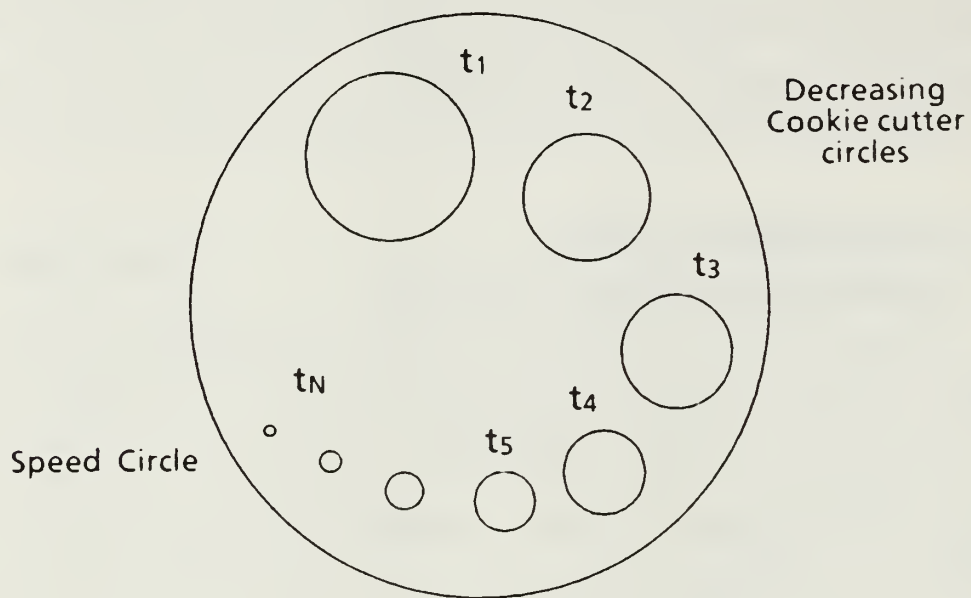
Figure 4.   Dips and Speed Circle.

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In his game, Danskin proposes that the area , A_0 be squeezed into a wedge of the same area in the speed circle. He solves the game and shows the value to be $\Gamma = \dfrac{A_0}{\pi U_{max}^2} = \dfrac{\theta}{2\pi}$ .This is the probability that the helicopter's wedge covers the point the submarine has picked in the speed circle given that the orientation of the wedge could be picked randomly.

Based on Danskin's assumptions, [Ref. 6] the helicopter has a better chance of detecting the submarine than in the flaming datum problem. The various cookie cutter sonar dips of the helicopter do not overlap each other. The submarine also does not employ any evasive maneuvers.

2. Dynamic Search Games

Thomas and Washburn [Ref. 9] have solved a very similar game which they termed 'dipping sonar' game. It contains all the elements of our flaming datum problem except that the rules of motion for the submarine are revised. In their game, the submarine is permitted to instantly choose any new position after each unsuccessful dip by the helicopter, as long as he stays within the Farthest - on - Circle (FOC), termed the "nimbleness" assumption. Another assumption made is that the submarine remains motionless in the speed circle between dips of the helicopter. All these assumptions are necessary for them to solve the game using Dynamic Programming. Briefly , the game is solved using the following recursive equation

$$Q(i,t) = \min_y \max_x \sum_j v_j \left\{ 1 - \sum_k x_k \, P(j, \; k, \; T(i,j,t)) \right\} \, Q(j, \; T(i,j,t)) \qquad (2.8)$$

where

- $Q(i,t)=$ value of the game , the probability that none of the remaining searches will detect the target if both sides play optimally. (i,t) represents the state of the game where the submarine is in cell, i , and time , t .
- $P(j,k,t)$ = Probability that a target in cell k will be detected by a search of cell j begun at time t .
- $T(i,j,t)=$ Time at which a search of cell j can begin if a search of cell i begins at time t .

The game proceeds in the following fashion:

1. After observing (i,t) the target will choose a cell k to hide while the searcher will choose a new cell j to search without knowing k.

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2. If $T(i,j,t) > t^*$, the termination time , then the target wins .
   Otherwise with probability $P(j, k, T(i,j,k))$, the target is detected and searcher wins.

3. If the target is not detected , set i to j and t to T(i,j,k) and return to 1.

The solution is obtained recursively with the end state $Q(i, t') = 1$ for $t' > t^*$ and with $y = (y_j)$ and $x = (x_k)$ being probability distribution over the cells for the searcher and the target respectively. The various positions to be picked by the helicopter and the submarine are obtained by dividing the speed circle into i cells of equal area. Applying the game to our flaming datum problem, this dynamic search game tends to favour the submarine on account of the nimbleness assumption. This favourable condition for the submarine might be neutralised by an additional assumption that the submarine is assumed motionless between dips. Intuitively, it seems that the nimbleness assumption would more than outweigh the motionless assumption.

III. MODELS, THEORY AND ASSUMPTIONS

A. ASSUMPTIONS

1. Submarine motion

The submarine is assumed to be moving always at a fixed speed U, once it starts its evasive maneuvers. Instantaneous changes in course are allowed.

2. Helicopter motion

The helicopter is also assumed to be moving at a constant speed V, during its search for the submarine. Acceleration and retardation are ignored.

3. Sonar Characteristics

The helicopter active sonar is assumed to illuminate a circular area of radius R, perfectly on each dip. Each dip can thus be considered to be a 'cookie cutter' where within such a radius, detection is certain and outside it, detection is impossible. Each sonar dip is assumed to take a constant time delay of D time units. This is attributed to time taken for the sonar device to be winched in and out of the water and time for the signal processing unit of the sonar to check for detection.

The submarine is assumed to know the latest position of the helicopter whenever it pings. However, this assumption was later relaxed to one in which only bearing information is known as we later found out that among the strategies explored, strategies that made use of this position information were dominated by those strategies which only make use of bearing information.

4. Unit speed circle

Danskin defines the speed circle to be a circle with constant radius U. Inside this circle, a 'cookie cutter' dip of the helicopter at time t , will have a radius of $\frac{R}{t}$. In our discussion, we will similarly use the concept of a unit speed circle which is defined to be a circle with constant unit radius. In our case, the relative size of a 'cookie cutter' sonar dip at time t, will have a radius of $\frac{R}{Ut}$.

5. Position of datum

The position of the datum is assumed to be accurately determined. This is definitely true when the flaming datum is still afloat but in the case that it is sunk, the last position is assumed to be accurately known by the helicopter.

B. STRATEGIES USED IN SIMULATION MODEL

1. Submarine Strategies

The submarine has two conflicting goals. One goal is to move away from the datum as fast as possible and hence expand the FOC rapidly. This action forces the helicopter to search in a bigger and bigger area hence reducing its search effectiveness. The other goal is to avoid the helicopter as much as it can by moving directly away from its last position or using some other avoidance strategy. This is especially important if the helicopter's last position (shown by its sonar dips) is near the submarine's position since the helicopter favours picking successive dips nearer to each other as they consume less of its limited mission time.

In the preliminary studies, several avoidance strategies were explored. One strategy involves making the submarine move perpendicular to the direction from the helicopter's last dip to its position at that time instant. The submarine which has the choice of two directions will pick the direction that will bring it further from the datum. Another strategy involves making the submarine move on a course that is almost perpendicular to the last two dips of the helicopter to ensure that it does not cut across the assumed path of the helicopter (using the direction of the last two dips as the helicopter course). Both of these strategies were found to be dominated by other strategies discussed and were thus discarded. In these cases, it seems that the submarine is making too many unnecessary changes of course and appears not to be moving much distance across the FOC. Another strategy has the submarine move directly towards the helicopter's last dip. This strategy was not used because it is difficult to establish the range to adopt this strategy. There is also a suspicion that this will only have a negligible effect on our results.

After some exploration, the number of strategies were reduced to the following two classes of simple avoidance strategies:

- Submarine moves directly away from the last dip.

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- Submarine ignores the last dip and proceeds directly away from the flaming datum. This strategy can be employed even if the search is passive.

It is also uncertain whether the submarine should be on the edge of the FOC at time τ_e, since the helicopter may search at the edge. The submarine can perhaps remain near the datum and start moving after the helicopter's first dip. Hence, we will also include r_2 as another parameter for the submarine to choose in its strategy where r_2 is the radial distance of the submarine from the datum in the speed circle at time τ_e. The submarine strategies are thus defined to be $s = (r_2, avoidance\ strategy)$.

2. Helicopter Strategy

In building the sets of strategies to be employed by the helicopter, an important principle is that the helicopter must not adopt any strategy which has a fixed pattern that can be exploited by the submarine. In Washburn's Expanding Area Search Experiment [Ref. 10], game participants found out that using fixed search patterns like spiralling inwards or outwards from the datum are not good strategies since they can be exploited by the submarine. Random movements are thus used to ensure that the submarine cannot exploit any of the helicopter's strategy.

From some of the analytical results discussed in related games, we saw that the helicopter should always attempt to distribute its search efforts as uniformly as possible over the entire FOC to be searched. This is done on the assumption that the submarine can be anywhere in the speed FOC. However this randomization of the positions of the dip is 'expensive' for the helicopter. The helicopter has a limited mission time and two dips placed far apart wil' consume much of this limited time . It will be more efficient for the helicopter to search within a localized area by carefully placing non-overlapping dips. This again may not be optimal since the submarine may be located at some distance away from the helicopter localized search area and thus cannot be detected at all. The helicopter has to compromise between conducting randomized search and localized search.

Another factor to consider is whether the helicopter should search on the edge of the FOC rather than the interior. If it is known that the submarine is always moving away from the datum, clearly the optimal strategy for the helicopter is also to search on the edges. The amount of edge searches to be used will also be included in the strategy of the helicopter.

To be precise, the helicopter at various times undertakes local searches that are either in the interior (inside) or on the edge of the FOC. The local searches are a succession of dips that are designed to be efficient at covering either its edge or its interior. To avoid the possibility of concentrating too much effort on a small part of the FOC , the helicopter occasionally abandons what he is doing and starts again. These moments when a new local search is started are called 'restarts'. The probability that a restart occurs after each dip is a constant ($P(Restart)$) that is part of the helicopter strategies. After each restart , the helicopter chooses a new interior point with probability, $P(interior)$, otherwise a new edge point , and begins making another local search until another restart. The helicopter's first local search is always of the interior type , and since the location of the first dip is especially important , r_1 (the radial location of the first dip in the speed circle) is also included as part of the helicopter strategy. Although an early dip will correspond to having a smaller FOC, some advantage will also be lost since the position is also near the edge and some of the area covered by the sonar detection area will be wasted outside the FOC. Using the unit speed circle, r_1 can vary from 0 to 1.0 . In our simulation model, pure strategies h for the helicopter are thus defined in terms of various combination of r_1, $P(Restart)$ and $P(interior)$.

$$h = (r_1, P(Restart), P(interior))$$

A simple way of determining the relationship of times between successive dips of the helicopter is to use the concept of the unit speed circle. If the helicopter chooses a new point (x,y) in the unit speed circle from its previous position (x,y) at time t, let t′ be the time that he arrives there. Travelling the physical distance from $(x,y)Ut$ to $(x',y')Ut'$ requires a time of $\dfrac{|(x,y)Ut - (x',y')Ut'|}{V}$. Therefore, t' must satisfy the equation

$$t + D + \frac{|(x,y)Ut - (x',y')Ut'|}{V} = t' \tag{3.1}$$

Equation (3.1) can be rearranged to obtain Equation (3.2), a quadratic equation in t'.

$$(t + D - t')^2 = |(x,y)\frac{U}{V}t - (x',y')\frac{U}{V}t'|^2 \tag{3.2}$$

Solving the equation (3.2), we have

$$t' = \left(\frac{d - abc + e}{1 - b^2} \right) t, \quad where$$

$$a = \sqrt{x^2 + y^2} \, \frac{U}{V}$$

$$b = \sqrt{x'^2 + y'^2} \, \frac{U}{V}$$

$$c = \text{cosine [angle between } (x,y) \text{ and } (x',y')]$$

$$d = 1 + \frac{D}{t}$$

$$e^2 = a^2 + b^2 d^2 - a^2 b^2 (1 - c^2) - 2abcd$$

Notice that $\frac{t'}{t}$ is independent of t when $D = 0$ and the relationship between t and t' is multiplicative.

C. GAME THEORY

To formulate our search problem as a two-person zero sum game, we define the pure strategies used by the helicopter to be $h = (r_1, P(Restart), P(interior))$ and that used by the submarine to be $s = (r_2, avoidance\ type)$ The payoff in this game is the probability of the submarine being detected , $P_d(h,s)$. The helicopter, being the maximizing player, will attempt to achieve

$$\max_h \min_s P_d(h,s)$$

While the submarine, being the minimizing player, will attempt to do the opposite by achieving

$$\min_s \max_h P_d(h,s)$$

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The value of the game v, is thus given by

$$\min_s \max_h \ P_d(h,s) \leq v \leq \ \max_h \min_s \ P_d(h,s) \tag{3.3}$$

D. MODIFIED RANDOM SEARCH EQUATION

In his Expanding Area Search Experiment, Washburn considers the case of an evader who knows that he is spotted at time t and maneuvers away at speed U to evade detection. The searcher starts the search after a time delay , τ_e using speed V and sweep width w (Detection occurs when the target is within $\frac{w}{2}$ from searcher). He shows that the probability of detection obtained in his experimental results is closely approximated by the equation

$$P_d = 1 \ - \ e^{-\int_{\tau_e}^{t}\left(\frac{Vw}{\pi U^2 \tau^2}\right)d\tau} \tag{3.4}$$

$$= 1 \ - \ e^{-\frac{Vw}{\pi U^2}\left[\frac{1}{\tau_e}-\frac{1}{t}\right]} \tag{3.5}$$

The assumptions used to derive the equation are
- Searcher searches randomly, which is a crucial assumption
- Detections in non-overlapping time intervals are independent

The formula is derived by reasoning that $\frac{Vwd\tau}{\pi U^2 \tau^2}$ is the ratio of area searched in dτ to area of farthest-on-circle at time τ , and is therefore the detection probability during the infinitisimal interval dτ . Summing over the entire search period we obtain the average number of detections in (τ_e, t) as

$$n(t) = \int_{\tau_e}^{t} \frac{Vwd\tau}{\pi U^2 \tau^2} = \frac{Vw}{\pi U^2}\left[\frac{1}{\tau_e}-\frac{1}{t}\right] \ \ for \ \ t \geq \ \tau_e \tag{3.6}$$

From the assumptions above, the number of detections in $(\tau_e,$t) is a Poission random variable and the probability of no detection is therefore equal to $e^{-n(t)}$. The probability of detection is therefore $1 - e^{-n(t)}$ as given in Equations (3.5).

Our flaming datum problem is different from Washburn's search experiment in that:

1. The searcher does not have a continuous search capability.

2. The target knows the position of the searcher whenever he makes a sonar dip.

We can derive a similar equation for our problem but modified to be

$$P_d = 1 - e^{-\int_{\tau_e}^{t} \frac{search\ rate}{FOC\ area} d\tau} \qquad (3.7)$$

$$= 1 - e^{-\int_{\tau_e}^{t} \frac{\frac{effective\ dip\ area}{(time\ per\ look)}}{FOC\ area} d\tau} \qquad (3.8)$$

The "time per look" represents the average amount of time between any two successive dips of the helicopter. It can be reasoned to be equal to $D + \dfrac{k_1 R}{V} + k_2(\dfrac{U\tau}{V})$. D represents the dipping time which is a constant in our game while k_1 and k_2 represent the amount of 'flying around' the helicopter makes in covering the FOC. Since $\dfrac{U\tau}{V}$ is the time taken to fly across half of the FOC, k_2 therefore measures the average amount of coverage of the entire FOC. The value of its upperbound is 2.0 which repesents the situation where the helicopter is always flying across and along the diameter of the FOC. k_1 is introduced because the helicopter must fly some fraction of its detection radius before making the next dip to avoid redundant coverage. The FOC area is given by $\pi U^2 \tau^2$ and the expression reduces to

$$P_d = 1 - e^{-\int_{\tau_e}^{t} \frac{\frac{effective\ dip\ area}{(D - k_1(\frac{R}{V}) - k_2(\frac{U}{V})\tau)}}{\pi U^2 \tau^2} d\tau} \qquad (3.9)$$

We also introduce k_3 to account for inefficiencies in covering the FOC area with circles, and k_4 because search will sometimes start after τ_e, the time the helicopter reaches the edge of the FOC. The final expression is thus given below:

$$P_d = 1 - e^{-\int_{\tau_e + k_4(\frac{R}{V})}^{t} \frac{\frac{k_3 \pi R^2}{(D - k_1(\frac{R}{V}) - k_2(\frac{U}{V})\tau)}}{\pi U^2 \tau^2} d\tau} \qquad (3.10)$$

where
- U - submarine speed
- V - helicopter speed

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- R - cookie cutter radius
- D - time delay in dipping
- $\tau_e = (\dfrac{L}{U+V})$ - time at which helicopter reaches edge of FOC.

The quantity

$$\frac{k_3 \pi R^2}{(D + k_1(\frac{R}{V})) + k_2(\frac{U}{V})\tau)}$$ (3.11)

is analogous to the search rate given by Vw in equations (3.4) to (3.6).

$n(t)$ in our modified equation is therefore given by

$$n(t) = \int_{\tau_e + k_4(\frac{R}{V})}^{t} \frac{k_3 \pi R^2}{\pi U^2 \tau^2 (D + k_1(\frac{R}{V})) + k_2(\frac{U}{V})\tau)} \, d\tau$$ (3.12)

Performing the integration, we get:

$$n(t) = \frac{k_3 R^2}{U^2(D + k_1(\frac{R}{V}))} \left[\frac{1}{\tau_e + k_4(\frac{R}{V})} - \frac{1}{t} \right]$$

$$- \frac{k_2 k_3}{U(D + k_1(\frac{R}{V}))^2} \log\left[\frac{t(D + k_1(\frac{R}{V})) + k_2(\frac{U}{V})(\tau_e + k_4(\frac{R}{V})))}{(\tau_e + k_4(\frac{R}{V}))[D + k_1(\frac{R}{V}) + k_2(\frac{U}{V})t]} \right]$$ (3.13)

The value of our search game should be approximately P_d if k_1, k_2, k_3 and k_4 are chosen to make equation (3.10) fit the simulation data as closely as possible.

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IV. METHODOLOGY

A. SIMULATION MODEL

Simulation models were used to determine the detection probability for each pair of helicopter-submarine strategies at each time t ($\tau_e \leq t < t^*$). Multiple programs were written since the submarine strategies cannot be easily generalised in a single program. Each program consisted of all the helicopter strategies and a single type of submarine strategy. The source codes are written in FORTRAN and implemented in the IBM 3033. The flow charts for the programs are found in appendix A. A total of 5000 runs are done for each program in order to obtain a precision of at least 10 % for small (0.04) probability of detection estimates. The coefficient of variation is given by

$$\frac{standard\ \ deviation}{mean} = \frac{\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}}{\hat{p}} = \sqrt{\frac{1-\hat{p}}{n\hat{p}}} \tag{4.1}$$

Each run is considered a Binomial trial with outcome $X_i = 1$ or 0 and n runs are conducted to get the probability of detection estimate

$$\hat{p} = \sum_{i=1}^{n} (\frac{X_i}{n}) \tag{4.2}$$

B. SOLVING THE GAME

The detection probabilities (or payoffs) from the various outputs of the various simulation models are read into a payoff matrix using a simple FORTRAN program for each time t ($t > \tau_e$). These payoff matrices for each time t are then solved sequentially using a linear programming subroutine DDLPRS available in New IMSL Library. If $x = (x_h)$ and $y = (y_s)$ represents the probability distribution for the helicopter and submarine strategies respectively, and v is the value of the game, the linear program can be formulated as

Minimize v subject to

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$$\sum_s y_s \, P_d(h,s) \leq v \, , \quad \forall \ h \tag{4.3}$$

$$\sum_s y_s = 1 \tag{4.4}$$

The primal of the LP solution gives the mixed strategies of the submarine while the dual solution gives the solution of the helicopter. Optimal strategies and the value of the game are obtained for given values of U, V, τ_e, R and t. A Flow chart for the program is also given in Appendix B.

C. MODEL FITTING

Although, the Random Search Equation (3.10) has five parameters, there are actually three dimensions in that expression. This can be easily seen by substituting $x_{Dummy} = \dfrac{U\tau}{R}$ into Equation (3.10) where we obtain the dimensions to be $\left(\tau_e + \dfrac{k_4 R}{V} \right)$. $\left(\dfrac{U}{V} \right)$ and $\left(\dfrac{DV}{R} \right)$. Hence, only $2^3 = 8$ combinations of dimensions are required to check if the simulation data could be fitted by the Random Search Equation. The values of the various game matrices obtained at each time t, are then used to estimate the constants k_1, k_2 and k_3 and k_4 . The curve fitting is done using a GRAFSTAT routine (non- linear regression curve fitting) available in NPS IBM 3033. Individual curves for each set of (τ_e, U, V, R, D) are fitted to get estimate of k_1 , k_2 . k_3 and k_4 . These curves are then combined to get one overall estimate for the unknown parameters. Statistical analysis is then performed to check the goodness of the fit.

D. COMPARISON WITH DYNAMIC SEARCH GAME MODEL

The values of the probability of detection obtained is then compared with the results given by the Dynamic Search Game by Thomas and Washburn [Ref. 9]. The results of the Dynamic search game was computed using a Fortran program developed by Washburn.

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V. RESULTS

A. TABLES

1. Strategies Used

A total of 10 submarine and 27 helicopter strategies were explored in the final stages of our experimentation. They are listed in Tables 1 and 2 respectively.

Table 1. SUBMARINE STRATEGIES USED

strategy	r_1	Avoidance type
y_1	0.2	Avoid helicopter
y_2	0.4	Avoid helicopter
y_3	0.6	Avoid helicopter
y_4	0.8	Avoid helicopter
y_5	1.0	Avoid helicopter
y_6	0.2	Avoid datum
y_7	0.4	Avoid datum
y_8	0.6	Avoid datum
y_9	0.8	Avoid datum
y_{10}	1.0	Avoid datum

Note

- Avoid helicopter refers to the strategy of moving directly away from the helicopter's last dip.

- Avoid Datum refers to moving directly away from datum (centre of FOC)

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Table 2. HELICOPTER STRATEGIES USED

strategy	r_1	P (Restart| search type)	P (interior search))
x_1	0.25	0.25	0.25
x_2	0.25	0.25	0.5
x_3	0.25	0.25	0.75
x_4	0.25	0.5	0.25
x_5	0.25	0.5	0.5
x_6	0.25	0.5	0.75
x_7	0.25	0.75	0.25
x_8	0.25	0.75	0.5
x_9	0.25	0.75	0.75
x_{10}	0.5	0.25	0.25
x_{11}	0.5	0.25	0.5
x_{12}	0.5	0.25	0.75
x_{13}	0.5	0.5	0.25
x_{14}	0.5	0.5	0.5
x_{15}	0.5	0.5	0.75
x_{16}	0.5	0.75	0.25
x_{17}	0.5	0.75	0.5
x_{18}	0.5	0.75	0.75
x_{19}	0.75	0.25	0.25
x_{20}	0.75	0.25	0.5
x_{21}	0.75	0.25	0.75
x_{22}	0.75	0.5	0.25
x_{23}	0.75	0.5	0.5
x_{24}	0.75	0.5	0.75
x_{25}	0.75	0.75	0.25
x_{26}	0.75	0.75	0.5
x_{27}	0.75	0.75	0.75

2. Parameters Used

The various sets of parameters used are given in the Table 3. This is the minimum number of sets of parameters required to check if the simulation model could

be approximated by the random search equation (3.10). Notice that normalized values are used where R and V are both equated to 1.0. This is done in order to simplify the expression used in the non-linear package.

Table 3.   PARAMETERS USED

U	V	τ_e	R	D
0.1	1	25.0	1	2
0.1	1	25.0	1	0
0.2	1	22.9	1	2
0.2	1	22.9	1	0
0.1	1	21.8	1	2
0.1	1	21.8	1	0
0.2	1	20.0	1	2
0.2	1	20.0	1	0

3.   Simulation Results

For a given set of of V,U, τ_e ,D and R, simulation results were obtained for the above combinations of submarine-helicopter strategies shown above. A sample output of one such result is shown in Table 5 of Appendix C. The result shows the probability of detection for each time t where $\tau_e \leq t < t^*$. A total of 270 data sets were generated for each set of parameter values.

4.   Game Results

The value of the game together with the optimal strategies was then obtained by solving each 27 x 10 game matrix for each time t, $(\tau_e \leq t < t^*)$. A sample output for a particular game is shown in Table 6 as Appendix D.   The entire process was repeated seven more times to obtain results for eight different sets of (U,V, τ_e,R,D) . A sample output is shown in Table 7 as Appendix E.

5.   Results of Curve fitting

The values obtained from the eight sets of parameter values are then used to estimate one   overall estimate of the unknown k_1, k_2, k_3 and k_4 of Equation (3.10).   The

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estimated k's as well as a statistical summary of the fit is shown in Table 4 below.

Table 4. STATISTICAL SUMMARY

```
                        TABLE OF COEFFICIENTS

628 OBSERVATIONS    R-SQUARED     = 0.99281   STANDARD ERROR = 0.018864
4 COEFFICIENTS      ADJ R-SQUARED = 0.99278   26 ITERATIONS

                                                      0.95 CONFIDENCE LIMITS
COEFFICIENT    ESTIMATE    STD ERR      T STAT     SIG LEVEL   LOWER         UPPER
K1             0.00021092  2.9340E-9    71889      0           0.00021091    0.00021093
K2             0.83762     1.1686E-5    71675      0           0.8376        0.83765
K3             1.0146      3.3727E-3      300.82   0           1.008         1.0212
K4             0.19498     2.5286E-6    77112      0           0.19498       0.19499
```

B. GRAPHS

1. Simulation data and Fitted Curves

The values of the games for the eight sets of (U, V, τ_e, R, D) are shown in Figure (5) and (6). The continous curves represent the curves fitted using non-linear regression while the symbols are actual data from the simulation results. Each curve represents a particular set of paramter values, and all the eight fitted curves use the same k_1, k_2, k_3 and k_4.
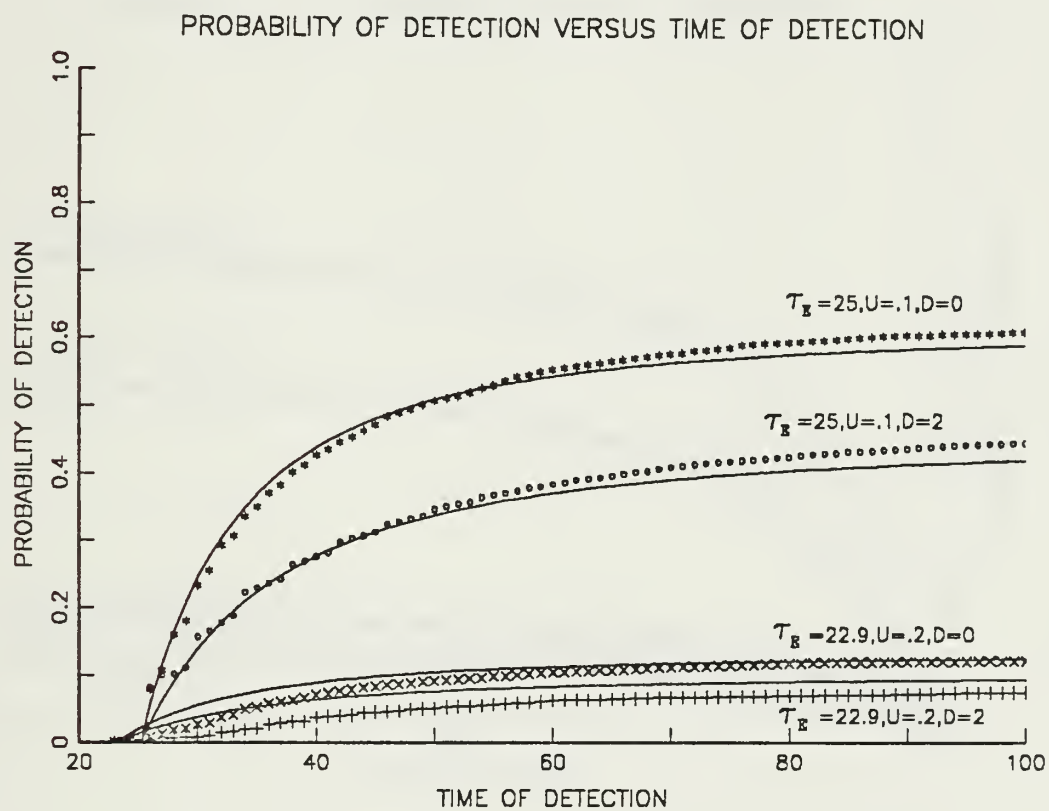
Figure 5.   Probability of Detection Curves
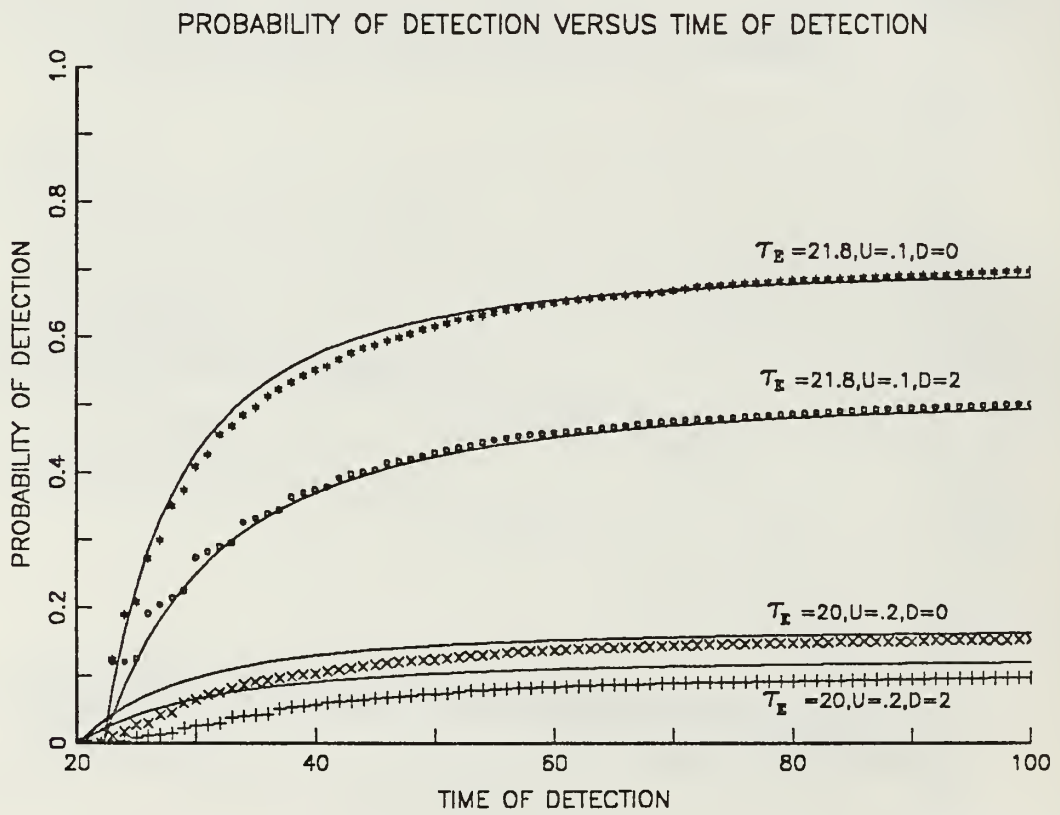
Figure 6.   Probability of Detection Curves

2. Random search versus Dynamic Search Model

In Figure (7) overleaf, the numbers represent the results from the Dynamic search model while the symbols shows the data from the simulation model. The fitted curves are also drawn in the figure. The parameters used are also shown in the legend just below all the curves. The four random search model curvès were just reproduced from Figures (5) and (6) above.

PROBABILITY OF DETECTION VERSUS TIME OF DETECTION

Figure 7.   Comparison of Results from Random Search and Dynamic Search Game
Models

VI. ANALYSIS OF RESULTS

A. STRATEGIES

1. Submarine

The results for the optimal strategies of the submarine show that the submarine generally favours staying near the edge of the FOC whenever it utilizes the 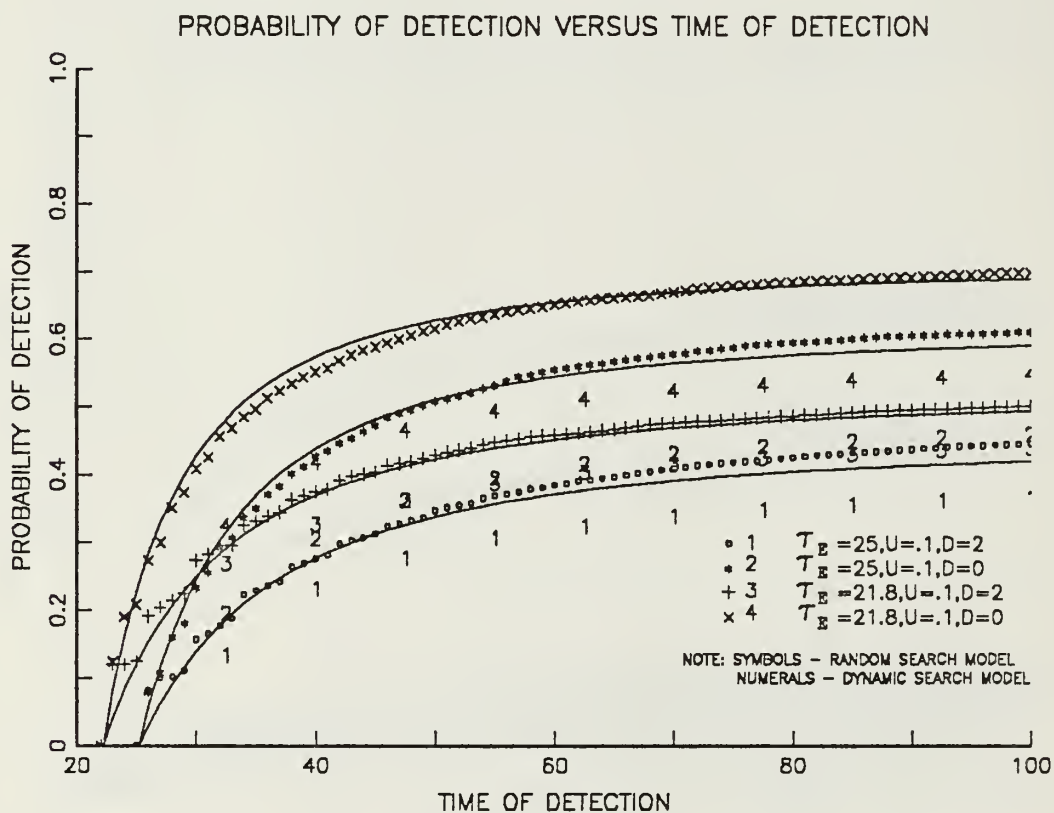class of strategies of moving away from the helicopter's last dip. The other class of strategies of ignoring the helicopter and just moving away from the datum is not often used. If it is used, the results shows that the submarine favours staying near the datum and will start moving away once the helicopter search begins.

2. Helicopter

The results show that the helicopter optimal strategies are mainly to have a higher porportion of localized interior search. It seems that too much 'flying around' is not optimal as only a limited number of dips can be conducted in the limited mission time of the helicopter. Also, it seems that much effort should be placed on searching the interior of the FOC, though some search must still be allocated to searching the edges so as to deter the submarine from staying at the edge of the FOC.

B. MODEL FITTING

The detection probability (or value of the game) was well estimated by the values obtained using the non-linear regression software in GRAFSTAT. The correlation coefficient is found to be close to 1 and there is no statistical evidence to indicate that these unknowns do not contribute to the equation at all.

These estimated values also give us an idea on the way optimal searches are conducted. k_1 is approximately zero which shows that the helicopter need not fly a minimum fraction of its detection radius to its next dip. The value of k_2 is about .83. This is fairly close to the mean distance of any two points in a circle as given by Kendall in his book, 'Geometric Probability' [Ref. 11]. The exact value of the mean was derived to be $(\frac{128R}{45\pi}) = .905R$. This shows that the helicopter was picking points fairly uniformly over the FOC. k_3 is almost equal to 1 and this shows that the helicopter was fairly effectively in covering the FOC area with its cookie cutter circles. This could be because most of the dips occur in the interior of the FOC. k_4 is approximately 0.2 which

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shows that most of the initial dips of the helicopter occurs near but not at the edge of the FOC.

C. COMPARISON WITH RESULTS OF DYNAMIC SEARCH GAME

The results given by the Dynamic search game gives a more pessimistic estimate of the probability of detection as compared to the simulation results. Generally, the results are about .07 to .15 lower. This shows that the 'nimbless' assumption used in the Dynamic search game outweighs the 'motionless between dips' assumption. There is also a greater difference when the value of D is zero. It is not clear which result is better since they do not differ very much.

D. WEAKNESS

The general weakness of this method is in not being able to investigate enough of the strategies. There are infinitely many strategies that can be used by the helicopter and submarine. However, many of them will tend to be dominated by certain classes of strategies. The assumptions used to derive the random search equation for the helicopter can again be criticized with the same argument as those used with any random search model. An example is the independent detections in non-overlapping intervals.

The submarine motion is assumed constant except for instantaneous changes in course. This assumption is not important since the submarine is always moving slowly and its turning radius is small relative to the detection radius of the active sonar. In the simulation, the submarine is able to determine the bearing of the dips accurately. This is obviously optimistic as to the direction finding capability of the submarine. The model could later be modified to account for transmission losses or other factors that will not provide such an accurate bearing of the sonar dips. The cookie cutter model is also a basic model for any detection system. Enhancement can perhaps be included but not much utility can be gained since the model itself is crude just like any other random search model.

VII.   PRACTICAL USES/APPLICATION

A.   COMPARISON OF AIRCRAFT AND HELICOPTER PERFORMANCES

The usefulness of such a technique is that it could be used together with other search games such as the Dynamic Search Games by Thomas and Washburn or The Helicopter Search Game by Danskin to compare aircraft platforms that have different speeds, different distances to datum and detection performances. For example, helicopters are normally located near the impending threat and are normally close to the flaming datum (if any) while aircraft like the P-3 Orion (carrying sonobouys) are usually located much further away but have greater speed. The question is which platform performs better.

VIII. CONCLUSION

The results of this experiment is yet another approximated solution to the two dimensional flaming datum problem. The random search equation (3.10) developed using the unknowns derived from the simulation can be used to compute a rough estimate of the search capability of a helicopter starting its search or it can similarly be used by a submarine to assess its probability of being detected. The computation can be done easily and quickly and it can also give us an idea of the various interactions of the basic parameters of any scenario. The results given by the Dynamic search game also support the simulations results. The attitude to be taken is that this is a practical tool to use until something better comes along or when general analytical techniques become available.

The exploration of various strategies used by both helicopter and submarine is also very useful as we discover that certain classes of strategies were always dominated by other classes. However, the optimal strategies obtained in the study cannot be taken as the 'true' optimal strategies to be used since there are infinitely many more to be explored. Rather, the results should provide us some idea of the distribution of search efforts (helicopter) or hiding efforts (submarine) especially when there are conflicting goals.

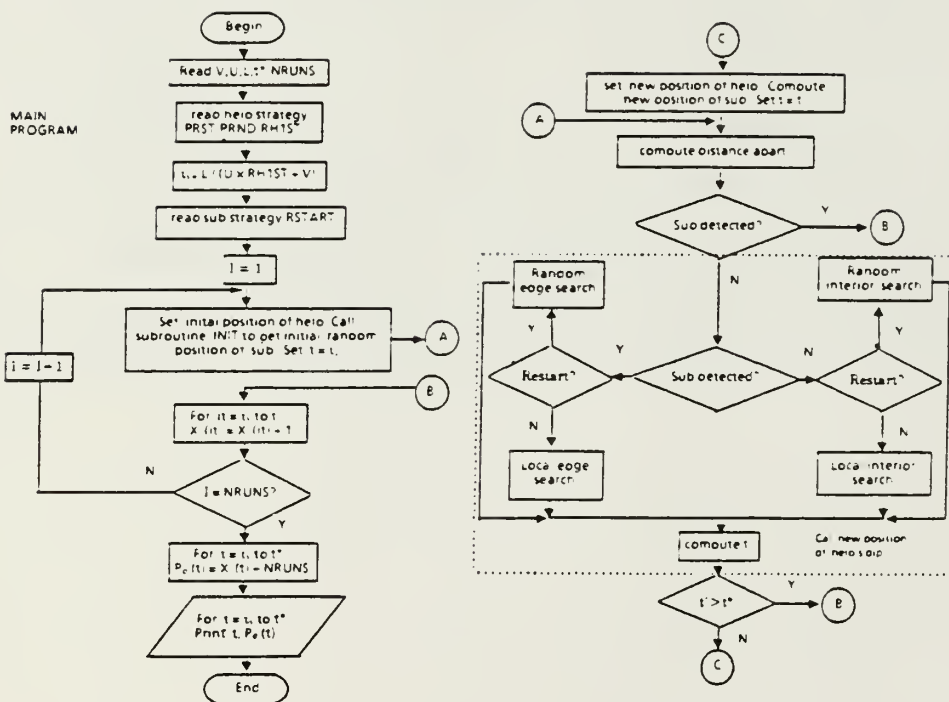# APPENDIX A.   FLOW CHART OF SIMULATION PROGRAM

Figure 8.   Flow chart for simulation program

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APPENDIX B.   FLOW CHART FOR SOLVING MATRIX GAME
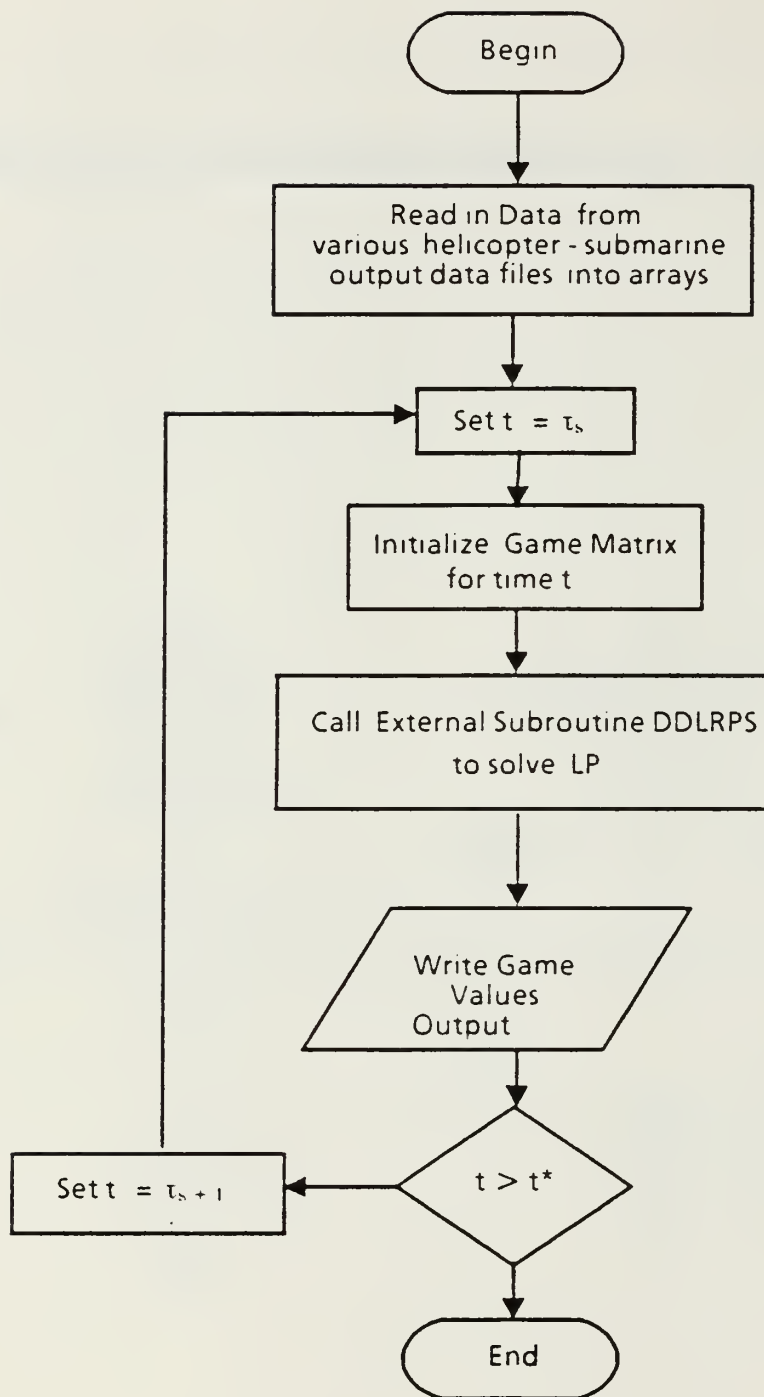
Figure 9.   Flow chart for solving matrix game using two person zero sum
           methodology

APPENDIX C.   SAMPLE OUTPUT FROM A SIMULATION MODEL

Table 5.   SAMPLE OUTPUT FROM A SIMULATION MODEL

```
0.000000000E+00
0.000000000E+00
0.000000000E+00
0.999999791E-02
0.399999991E-01
0.799999833E-01
0.899999738E-01
0.999999642E-01
0.129999995
0.139999986
0.159999967
0.169999957
0.169999957
0.179999948
0.209999979
0.219999969
0.229999959
0.250000000
0.259999990
0.269999981
0.269999981
0.279999971
0.279999971
0.289999962
0.309999943
0.309999943
0.309999943
0.309999943
0.319999993
0.319999993
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0.339999974
```

PRST	0.750000000	PAREA 0.750000000	R	1.00000000
1.00000000	0.100000024	20.0000000	0.000000000E+00
100.000000			
1.00000000	27.5000000	0.250000000	0.000000000E+00
26.8292694			

Note: The output above are the detection probabilities obtained for each time t $(25 \leq t < 100)$ when the helicopter utilizes strategy x_9 against the submarine strategy y_5 The time of the first dip was computed to be 26.829. This is because the initial dip was inside the FOC.

The parameters used in this model are given below:

- $V = 1$, $U = .1$, $D = 0$, $t^* = 100$, $L = 27.5$, $R = 1.0$, $r_1 = 0.25$, $r_2 = 1.0$,

APPENDIX D. SAMPLE OUTPUT AFTER SOLVING A GAME

Table 6. SAMPLE OUTPUT OF VALUE AND OPTIMAL STRATEGIES OF
GAME WITH TIME

```
TIME            50

OBJECTIVE =     0.5218

PRIMAL
Y1      Y2      Y3      Y4      Y5      Y6      Y7      Y8      Y9      Y10
0.0000  0.0994  0.0000  0.7453  0.1553  0.0000  0.0000  0.0000  0.0000  0.0000


X1      X2      X3      X4      X5      X6      X7      X8      X9
0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000


X10     X11     X12     X13     X14     X15     X16     X17     X18
0.0000  0.0000  0.2112  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000


X19     X20     X21     X22     X23     X24     X25     X26     X27
0.0000  0.0000  0.3975  0.0000  0.0000  0.3913  0.0000  0.0000  0.0000
```

Note: The output is for the case when time of detection is 50 time units. The parameters used in this model is given below:

- $V = 1$
- $U = .1$
- $D = 0$
- $t^* = 100$
- $R = 1.0$
- $L = 27.5$ giving $\tau_e = 25$

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APPENDIX E. SAMPLE OUTPUT OF GAME VALUES VERSUS TIME

Table 7. SAMPLE OUTPUT OF GAME VALUES OBTAINED

Time of Detection	Value (Prob. of Detection)	Time of Detection	Value (Prob. of Detection)	Time of Detection	Value (Prob. of Detection)
25	0.0000	51	0.5243	77	0.5833
26	0.0849	52	0.5307	78	0.5872
27	0.1157	53	0.5350	79	0.5890
28	0.1600	54	0.5357	80	0.5891
29	0.1800	55	0.5362	81	0.5891
30	0.2558	56	0.5362	82	0.5900
31	0.2676	57	0.5362	83	0.5919
32	0.3176	58	0.5405	84	0.5921
33	0.3276	59	0.5407	85	0.5933
34	0.3650	60	0.5407	86	0.5976
35	0.3727	61	0.5448	87	0.5976
36	0.3969	62	0.5490	88	0.5976
37	0.4151	63	0.5491	89	0.5994
38	0.4298	64	0.5499	90	0.5994
39	0.4454	65	0.5575	91	0.5994
40	0.4540	66	0.5653	92	0.6040
41	0.4596	67	0.5686	93	0.6040
42	0.4679	68	0.5725	94	0.6082
43	0.4787	69	0.5734	95	0.6089
44	0.4803	70	0.5753	96	0.6089
45	0.4837	71	0.5753	97	0.6104
46	0.4963	72	0.5753	98	0.6104
47	0.4971	73	0.5754	99	0.6108
48	0.5059	74	0.5761	100	0.6108
49	0.5150	75	0.5761		
50	0.5218	76	0.5767		

Note:

- $V = 1, U = .1, \tau_e = 25, D = 0, R = 1.$

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